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Energy scales in a stabilized brane world

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Abstract

Brane world gravity looks different for observers on positive and negative tension branes. First we consider the well-known RS1 model with two branes embedded into the AdS_5 space-time and recall the results on the relations between the energy scales for an observer on the negative tension brane, which is supposed to be "our" brane. Then from the point of view of this observer we study energy scales and masses for the radion and graviton excitations in a stabilized brane world model. We argue that there may be several possibilities leading to scales of the order 1-10 TeV or even less for new physics effects on our brane. In particular, an interesting scenario can arise in the case of a "symmetric" brane world with a nontrivial warp factor in the bulk, which however takes equal values on both branes.

Brane world models with extra space dimensions and different fields living in the bulk and on the branes present a large number of intriguing possibilities to solve or to address from a new viewpoint various problems in particle physics and cosmology. These are the hierarchy problem, the proton stability, the baryogenesis and leptogenesis, the small masses and large mixing of neutrinos, the dark matter and dark energy etc. It was demonstrated that such models can be related to the string theories and can incorporate sypersymmetry and other possible symmetries in a natural way. Moreover, the brane world models lead to very interesting predictions for experiments at TeV energy colliders, - Tevatron, LHC, and ILC. [1]

In brane world scenarios the crucial point is the energy scales involved in the model, which are usually defined by the gravitational interaction. This is a reflection of the fact that it is gravity that forms the space-time framework for such models. In the present paper we are going to study the energy scales rendered by the gravitational interaction in stabilized brane word scenarios.

We start with considering the simplest well-known case with two branes embedded into the AdS_5 space-time, - the RS1 model. It is based on the exact solution for a system of two branes interacting with gravity in a five-dimensional space-time, which was found in paper [2]. The theory of linearized gravity interactions based on this solution is called the Randall-Sundrum model (usually abbreviated as RS1 model), and it is widely discussed in the literature. An apparent flaw of this model is the presence of a massless scalar mode

[1]It is almost impossible to give references to a very large number of papers in this field. We refer to the recent this year reviews [1] where original references can be found.

called the radion, whose interactions may contradict the existing experimental data. This feature of the model reflects the fact that the brane separation distance is not fixed in it, but remains an arbitrary parameter.

An approach to curing this flaw was put forward in paper [3], where it was shown that a bulk scalar field with an appropriate interaction can stabilize the brane separation distance. In this case the radion should acquire a mass, which is defined by the backreaction of the scalar field on the metric. Though the latter was not considered in paper [3].

An exact solution for a system of two branes interacting with gravity and a scalar field in a five-dimensional space-time was found in [4]. This solution allows one to take into account the backreaction of the scalar field on the metric and thus to find the radion mass. Under certain assumptions the radion mass was computed in paper [5].

Although the solution for the metric in [4] has more parameters than the original RS1 solution, it is usually tacitly implied that the parameters coinciding with those of the RS1 model take the same values and that the five- and four-dimensional energy scales are still related by the formulas of the RS1 model.

In the present paper we study this problem in more detail and show that there are other possibilities of choosing the parameters in addition to those suggested by the RS1 model. To this end we reconsider first the relations between the five- and four-dimensional energy scales in the RS1 model. We follow the methods and the notations of paper [2].

Let us denote the coordinates of five-dimensional space-time by $\{x^M\} \equiv \{x^\mu, y\}$, $M = 0,1,2,3,4$, $\mu = 0,1,2,3$, the coordinate $x^4 \equiv y$, $\quad -L \le y \le L$ parameterizing the fifth dimension. It forms the orbifold S^1/Z_2, which is realized as the circle of the circumference $2L$ with the points y and $-y$ identified (thus, y is the linear coordinate on the circle, which is related to the angular coordinate ϕ used in [2] by $y = \pi\phi$, in particular $L = \pi r_c$). Correspondingly, the metric g_{MN} satisfies the orbifold symmetry conditions

$$\begin{aligned} g_{\mu\nu}(x,-y) &= g_{\mu\nu}(x,y), \\ g_{\mu 4}(x,-y) &= -g_{\mu 4}(x,y), \\ g_{44}(x,-y) &= g_{44}(x,y). \end{aligned} \tag{1}$$

The branes are located at the fixed points of the orbifold, $y = 0$ and $y = L$.

The action of the model is

$$S = S_g + S_b, \tag{2}$$

where S_g and S_b are given by

$$\begin{aligned} S_g &= \int d^4x \int_{-L}^{L} dy \left(2M^3 R - \Lambda\right) \sqrt{-g}, \\ S_b &= -\lambda_1 \int_{y=0} \sqrt{-\tilde{g}} d^4x - \lambda_2 \int_{y=L} \sqrt{-\tilde{g}} d^4x. \end{aligned} \tag{3}$$

Here $\tilde{g}_{\mu\nu}$ is the induced metric on the branes and the subscripts 1 and 2 label the branes. We also note that the signature of the metric g_{MN} is chosen to be $(-,+,+,+,+)$. The Randall-Sundrum solution for the metric is given by

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \equiv \gamma_{MN}(y) dx^M dx^N, \tag{4}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and the function

$$\sigma(y) = k|y| + c \tag{5}$$

in the interval $-L \le y \le L$. It is well known that the constant c in the definition of σ can be eliminated by rescaling the coordinates $\{x^\mu\}$ in (4). Nevertheless, we will keep it, because it will be useful for our considerations.

The parameter k is positive and has the dimension of mass, the bulk cosmological constant Λ and the brane tensions $\lambda_{1,2}$ are defined by the fine-tuning conditions:

$$\Lambda = -24M^3k^2, \quad \lambda_1 = -\lambda_2 = 24M^3k. \tag{6}$$

We see that the brane at $y = 0$ has a positive energy density, whereas the brane at $y = L$ has a negative one.

The linearized theory in the RS1 background is obtained by representing the metric as

$$g_{MN} = \gamma_{MN} + h_{MN}, \tag{7}$$

substituting it into action (2) and keeping only the terms quadratic in h_{MN} [6]. The corresponding equations of motion can be decoupled [6] to give an equation for a transverse-traceless tensor field $b_{\mu\nu}(x, y)$, which describes the graviton and its massive excitations, and a massless radion scalar field $\phi(x)$. The equation for the field $b_{\mu\nu}(x, y)$ looks like

$$\frac{1}{2}(e^{2\sigma(y)}\Box b_{\mu\nu} + \frac{\partial^2 b_{\mu\nu}}{\partial y^2}) - b_{\mu\nu}(2(\sigma')^2 - \sigma'') = 0 \tag{8}$$

and defines the mass spectrum of the tensor particles (here and below $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the D'Alembert operator and the prime denotes the derivative with respect to y). It is a matter of simple calculation to check that the massless modes of this equation have the form

$$b_{\mu\nu}(x, y) = e^{-2\sigma(y)}\bar{h}_{\mu\nu}(x). \tag{9}$$

In order to derive a relation between the four- and five dimensional energy scales, following paper [2] we consider the metric only with the zero mode excitations

$$ds^2 = e^{-2\sigma(y)}(\eta_{\mu\nu} + \bar{h}_{\mu\nu}(x))dx^\mu dx^\nu + dy^2 = e^{-2\sigma(y)}\bar{g}_{\mu\nu}(x)dx^\mu dx^\nu + dy^2, \tag{10}$$

where we introduced the notation

$$\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \bar{h}_{\mu\nu}(x). \tag{11}$$

If we substitute this metric into action (2), we get the following result:

$$S = 2M^3 \int_{-L}^{L} e^{-2\sigma(y)}dy \int d^4x R_4(\bar{g})\sqrt{-\bar{g}}, \tag{12}$$

$R_4(\bar{g})$ being the four-dimensional scalar curvature. Integrating in (12) over the extra dimension, we get an effective action

$$S_{eff} = 2M^3 e^{-2c}\frac{1 - e^{-2kL}}{k}\int d^4x R_4(\bar{g})\sqrt{-\bar{g}}. \tag{13}$$

Thus, we see that the reduced action is the standard gravitational action, and therefore the field $\bar{h}_{\mu\nu}(x)$ is the graviton field.

The expression in front of the integral is the coupling constants of the graviton field to matter on the branes. Since we have two branes, we have to find out explicitly, to which brane this coupling constant corresponds.

The coupling of gravity to matter on any brane is given by

$$\int d^4x\sqrt{-\tilde{g}}L(\phi,\tilde{g}), \tag{14}$$

where $\tilde{g}$ is the metric induced on the brane and ϕ denotes an arbitrary set of fields. An important point is that in order to get the canonical energy-momentum tensor, the Lagrangian L must have the canonically normalized kinetic terms. For example, in the case of ϕ being a scalar field it has the form $-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

Linearizing interaction (14), we get

$$\begin{aligned}\int d^4x\sqrt{-\gamma}\left(\frac{\delta L}{\delta\gamma_{\mu\nu}}-\frac{1}{2}\gamma^{\mu\nu}L\right)\delta\tilde{g}_{\mu\nu} = \frac{1}{2}\int d^4x\sqrt{-\gamma}e^{-2\sigma}\bar{h}_{\mu\nu}T^{\mu\nu} = \\ = \frac{1}{2}\int d^4xe^{-2\sigma}\bar{h}_{\mu\nu}T_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma}.\end{aligned} \tag{15}$$

Let us first consider the interaction with matter on the positive tension brane. Putting $y=0$ in (5) and (15), we get

$$ds^2 = e^{-2c}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 = \gamma_{\mu\nu}(0)dx^\mu dx^\nu + dy^2, \tag{16}$$

and the interaction with matter of the form

$$\frac{1}{2}\int d^4xe^{-2c}\bar{h}_{\mu\nu}T_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma}. \tag{17}$$

We recall (see [7]) that the coordinates are called Galilean, if $\gamma_{\mu\nu} = diag(-1,1,1,1)$. Though physics does not depend on the choice of the coordinates, this is the preferred coordinate system, in which we interpret experimental results and where the space coordinates are measured by a "ruler", the time is measured by a "clock" and all the physical observables are measured in the standard units (cm, GeV, etc.). Equation (16) implies that the coordinates $\{x^\mu\}$ in (4) are Galilean on the brane at 0 only if $c=0$. In this case equation(13) gives the well-known relation between the energy scale M_P on the positive tension brane and in the bulk [2]:

$$M_P^2 = M^3\frac{1-e^{-2kL}}{k}. \tag{18}$$

Since for an observer on the positive tension brane M_P is equal to the four-dimensional gravitational energy scale on this brane, he has to conclude that $M\sim k$ are also of the same order. Thus, for an observer on the positive tension brane the fundamental five-dimensional energy scale is of the order of the four-dimensional gravitational energy scale on this brane.

Now let us consider the interaction with matter on the negative tension brane. Putting $y=L$ in (5) and (15), we get

$$ds^2 = e^{-2(kL+c)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 = \gamma_{\mu\nu}(L)dx^\mu dx^\nu + dy^2, \tag{19}$$

and the interaction Lagrangian

$$\frac{1}{2}\int d^4x e^{-2(kL+c)}\bar{h}_{\mu\nu}T_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma}. \tag{20}$$

Now equation (19) gives that the coordinates $\{x^\mu\}$ in (4) are Galilean on the brane at L only if $c = -kL$. Substituting this value into equation (13) gives the relation between the energy scale M_N on the negative tension brane and in the bulk [6, 8, 9]:

$$M_N^2 = M^3\frac{e^{2kL}-1}{k}. \tag{21}$$

For an observer on the negative tension brane $M_N = M_{PL}$, which can be achieved by taking $M \sim k \sim 1TeV$ and $kL \simeq 35$. Thus, for an observer on the negative tension brane the physical picture is quite different from that of an observer on the positive tension brane. The fundamental five-dimensional energy scale can lie in the TeV region, whereas the four-dimensional energy scale, explaining the weakness of the gravitational interaction, arises due to the warp factor in the metric.

Thus, we see that the gravity in the bulk looks different, when viewed from different branes. In the case of the negative tension brane one can speak of a "bottom-up" picture, when a large energy scale arises from a small one due to the warp factor. In other words, the fundamental energy scale can be of the TeV order and the four-dimensional Planck scale on our brane appears as an effective scale $M_{PL} = Me^{kL}$, which explains the observed weakness of Newtonian gravity.

Now let us consider a stabilized brane world model. To be specific, we take the model admitting an exact solution found in [4]. Its action can be written as

$$S = S_g + S_\phi, \tag{22}$$

where S_g and S_ϕ are given by

$$\begin{aligned} S_g &= 2M^3\int d^4x\int_{-L}^{L} dyR\sqrt{-g}, \qquad (23)\\ S_\phi &= -\int d^4x\int_{-L}^{L} dy\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right)\sqrt{-g} - \\ &- \int_{y=0}\sqrt{-\tilde{g}}\lambda_1(\phi)d^4x - \int_{y=L}\sqrt{-\tilde{g}}\lambda_2(\phi)d^4x. \end{aligned}$$

Here $V(\phi)$ is a bulk scalar field potential and $\lambda_{1,2}(\phi)$ are brane scalar field potentials. They will be specified later.

The standard ansatz for the metric and the scalar field, which preserves the Poincaré invariance in any four-dimensional subspace $y = const$ looks like

$$ds^2 = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \equiv \gamma_{MN}(y)dx^M dx^N, \tag{24}$$

$$\phi = \phi(y). \tag{25}$$

If one substitutes this ansatz into action (22), one gets a rather complicated system of nonlinear differential equations for functions $A(y), \phi(y)$ [4]. If the bulk potential $V(\phi)$ can be represented as

$$V(\phi) = \frac{1}{8}\left(\frac{\delta W}{\delta \phi}\right)^2 - \frac{1}{24M^3}W^2(\phi),$$

the original system in the bulk is equivalent to a pair of equations

$$\phi'(y) = \frac{1}{2}\frac{\delta W}{\delta \phi}, \quad A'(y) = \frac{1}{24M^3}W(\phi), \tag{26}$$

which can be solved exactly for an appropriate choice of the function $W(\phi)$. To get a solution of the complete system, one has to fine-tune the brane scalar field potentials, which is similar to fine-tuning the cosmological constant and the brane tensions in the RS1 model.

To obtain a quartic potential $V(\phi)$, the function W can be chosen as

$$W(\phi) = 24M^3k - u\phi^2, \tag{27}$$

where the parameters k and u have the dimension of mass. Then the fine-tuned brane potentials must be of the form

$$\begin{aligned} \lambda_1(\phi) &= W(\phi_0) + W'(\phi_0)(\phi - \phi_0) + \beta_1^2(\phi - \phi_0)^2, \\ \lambda_2(\phi) &= -W(\phi_L) - W'(\phi_L)(\phi - \phi_L) + \beta_2^2(\phi - \phi_L)^2. \end{aligned} \tag{28}$$

The parameter k is, in fact, the standard parameter of the RS1 model, which is related to the bulk cosmological constant $\Lambda = V(0)$ by the same formula, as in (6). Thus, the parameters of the model are $k, u, \phi_0, \phi_L, \beta_{1,2}$, as well as the fundamental five-dimensional gravitational scale M. When the former parameters are made dimensionless with the help of M, they should be of the order $O(1)$, so that there is no hierarchical difference in the parameters.

For such a choice of the potentials, the solutions for functions $A(y), \phi(y)$ look like

$$\begin{aligned} \phi(y) &= \phi_0 e^{-u|y|}, \\ A(y) &= k|y| + \frac{\phi_0^2}{48M^3}e^{-2u|y|} + c. \end{aligned} \tag{29}$$

In the sequel we will denote

$$b = \frac{\phi_0^2}{48M^3}, \tag{30}$$

and write $A(y)$ as

$$A(y) = k|y| + be^{-2u|y|} + c. \tag{31}$$

Here again we keep the constant c, which can be removed by a rescaling of the coordinates $\{x^\mu\}$ in metric (24). Now in order to have Galilean coordinates one has to take $c = -b$ for the positive tension brane and $c = -kL - be^{-2uL}$ for the negative tension brane.

The separation distance in defined by the equation

$$L = \frac{1}{u}\ln\frac{\phi_0}{\phi_L} \tag{32}$$

and, therefore, it is stabilized. Keeping this in mind, it is convenient to use L instead of ϕ_L as a parameter of the theory.

Now the linearized theory is obtained by representing the metric and the scalar field as

$$
\begin{aligned}
g_{MN}(x,y) &= \gamma_{MN}(y)+h_{MN}(x,y), && (33)\\
\phi(x,y) &= \phi_0 e^{-u|y|}+f(x,y), && (34)
\end{aligned}
$$

substituting this representation into action (22) and keeping the terms of the second order in h_{MN} and f. The corresponding equations of motion are:

1. $\mu\nu$-component

$$
\begin{gathered}
\tfrac{1}{2}(\partial_\sigma\partial^\sigma h_{\mu\nu}-\partial_\mu\partial^\sigma h_{\sigma\nu}-\partial_\nu\partial^\sigma h_{\sigma\mu}+\partial_4\partial_4 h_{\mu\nu})+\tfrac{1}{2}\partial_\mu\partial_\nu\tilde{h}+\tfrac{1}{2}\partial_\mu\partial_\nu h_{44} \quad - \\
-\tfrac{1}{2}\partial_4(\partial_\mu h_{4\nu}+\partial_\nu h_{4\mu})+A'(\partial_\mu h_{4\nu}+\partial_\nu h_{4\mu})+\tfrac{1}{2}\gamma_{\mu\nu}\Big(-\partial_4\partial_4\tilde{h}- \\
-\partial_\sigma\partial^\sigma h_{44}-\partial_\sigma\partial^\sigma\tilde{h}+4A'\partial_4\tilde{h}-3A'\partial_4 h_{44}+\partial_\sigma\partial_\tau h^{\sigma\tau}+2\partial^\sigma\partial_4 h_{\sigma 4}- \\
-4A'\partial^\sigma h_{4\sigma}\Big)-h_{\mu\nu}(2A'^2-A'')+\tfrac{3}{2}h_{44}\gamma_{\mu\nu}(4A'^2-A'')- \\
-\tfrac{1}{4M^3}\left(\gamma_{\mu\nu}f'\phi'-f\gamma_{\mu\nu}(4A'\phi'-\phi'')\right)=0; \qquad (35)
\end{gathered}
$$

2. $\mu 4$-component

$$
\begin{gathered}
\partial_4(\partial_\mu\tilde{h}-\partial^\nu h_{\mu\nu})+\partial^\nu(\partial_\nu h_{\mu 4}-\partial_\mu h_{\nu 4})+3A'\partial_\mu h_{44}+ \qquad (36)\\
+\frac{1}{2M^3}\partial_\mu f\phi'=0;
\end{gathered}
$$

3. 44-component

$$
\begin{gathered}
\partial^\mu(\partial^\nu h_{\mu\nu}-\partial_\mu\tilde{h})-6A'\partial^\mu h_{\mu 4}+3A'\partial_4\tilde{h}+ \qquad (37)\\
+\frac{1}{2M^3}\left(-h_{44}V-f\frac{\delta V}{\delta\phi}+f'\phi'\right)=0;
\end{gathered}
$$

4. equation for the field f

$$
\begin{gathered}
h_{44}\left(\tfrac{\delta\lambda_1}{\delta\phi}\delta(y)+\tfrac{\delta\lambda_2}{\delta\phi}\delta(y-L)\right)-2h_{44}(-4A'\phi'+\phi'')+ \\
+\phi'\partial_4\tilde{h}-\phi'\partial_4 h_{44}-8A'\partial_4 f+2\partial_M\partial^M f-2\phi'\partial^\mu h_{\mu 4}- \\
-2f\left(\tfrac{\delta^2 V}{\delta\phi^2}+\tfrac{\delta^2\lambda_1}{\delta\phi^2}\delta(y)+\tfrac{\delta^2\lambda_2}{\delta\phi^2}\delta(y-L)\right)=0. \qquad (38)
\end{gathered}
$$

Here and below ϕ stands for the background solution (29) and $\tilde{h}=\gamma^{\mu\nu}h_{\mu\nu}$. We will also use the following auxiliary equation, which is obtained by contracting the indices in the $\mu\nu$-equation:

$$
\begin{gathered}
\partial^\mu\partial^\nu h_{\mu\nu}-\partial^\mu\partial_\mu(\tilde{h}+\tfrac{3}{2}h_{44})-6A'\partial_4(h_{44}-\tilde{h})-\tfrac{3}{2}\partial_4\partial_4\tilde{h}-6A'\partial^\mu h_{\mu 4}+ \\
+3\partial^\mu\partial_4 h_{\mu 4}+6h_{44}(4A'^2-A'')-\tfrac{1}{M^3}(f'\phi'+f(-4A'\phi'+\phi''))=0. \qquad (39)
\end{gathered}
$$

These equations are invariant under the gauge transformations

$$
\begin{aligned}
h'_{\mu\nu}(x,y) &= h_{\mu\nu}(x,y) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\gamma_{\mu\nu}\partial_4 A \xi_4), \\
h'_{\mu 4}(x,y) &= h_{\mu 4}(x,y) - (\partial_\mu \xi_4 + \partial_4 \xi_\mu + 2\partial_4 A \xi_\mu), \\
h'_{44}(x,y) &= h_{44}(x,y) - 2\partial_4 \xi_4, \qquad (40) \\
f'(x,y) &= f(x,y) - \partial_4 \phi \xi_4, \quad \partial_4 \equiv \partial/\partial y, \qquad (41)
\end{aligned}
$$

provided $\xi_M(x,y)$ satisfy the orbifold symmetry conditions

$$
\xi_\mu(x,-y) = \xi_\mu(x,y), \quad \xi_4(x,-y) = -\xi_4(x,y).
$$

These gauge transformations are a generalization of the gauge transformations in the unstabilized RS1 model [6, 8]. We will use them to isolate the physical degrees of freedom of the fields h_{MN} and f.

The gauge transformations with function ξ_4 allow one to impose the gauge condition

$$
(e^{-2A}h_{44})' - \frac{1}{3M^3}e^{-2A}\phi' f = 0. \qquad (42)
$$

This relation was obtained in [5] from the equation for $\mu 4$-component, in which only the scalar degrees of freedom were retained.

Similar to the case of the unstabilized RS1 model, the gauge transformations with functions ξ_μ allow one to impose the gauge $h_{\mu 4}(x,y) = 0$. After which there remain the gauge transformations satisfying

$$
\partial_4(e^{2A}\xi_\mu) = 0. \qquad (43)
$$

Thus, we can use the gauge

$$
\begin{aligned}
(e^{-2A}h_{44})' - \frac{1}{3M^3}e^{-2A}\phi' f &= 0, \qquad (44) \\
h_{\mu 4} &= 0.
\end{aligned}
$$

Next we represent the gravitational field as

$$
h_{\mu\nu} = b_{\mu\nu} + \frac{1}{4}\gamma_{\mu\nu}\tilde{h}, \qquad (45)
$$

with $b_{\mu\nu}$ being a traceless tensor field ($\gamma^{\mu\nu}b_{\mu\nu} = 0$).

Substituting gauge conditions (44) and representation (45) into the $\mu 4$-equation and into contracted $\mu\nu$-equation (39), we get:

$$
\begin{aligned}
-\partial_4(\partial^\nu b_{\mu\nu}) + \tfrac{3}{4}\partial_\mu \partial_4(\tilde{h} + 2h_{44}) &= 0, \qquad (46) \\
\partial^\mu \partial^\nu b_{\mu\nu} - \tfrac{3}{4}\partial_\rho \partial^\rho \tilde{h} - \tfrac{3}{2}\partial_\rho \partial^\rho h_{44} - \tfrac{3}{2}\tfrac{\partial^2}{\partial y^2}\tilde{h} + & \\
+6A'\partial_4 \tilde{h} - 3\tfrac{\partial^2}{\partial y^2}h_{44} + 12A'\partial_4 h_{44} &= 0. \qquad (47)
\end{aligned}
$$

Equation (46) suggest the substitution $\tilde{h} = -2h_{44}$, which allows one to decouple the equations for the fields $b_{\mu\nu}, h_{44}$ and f. Really, as a result of this substitution equations (46), (47) take the form

$$
\begin{aligned}
\partial_4(\partial^\nu b_{\mu\nu}) &= 0, \qquad (48) \\
\partial^\mu \partial^\nu b_{\mu\nu} &= 0. \qquad (49)
\end{aligned}
$$

It is not difficult to check that the residual gauge transformations (43) are sufficient to impose the gauge

$$\partial^\nu b_{\mu\nu} = 0,$$

in which the former equations are satisfied identically.

Thus, in what follows we will be working in the gauge

$$\begin{aligned}(e^{-2A}h_{44})' - \frac{1}{3M^3}e^{-2A}\phi' f = 0,\\ h_{\mu 4} = 0,\\ \partial^\nu b_{\mu\nu} = 0,\end{aligned} \tag{50}$$

the residual gauge transformations now being

$$\xi_\mu = e^{-2A}\epsilon_\mu(x), \qquad \partial^\nu \epsilon_\nu(x) = 0, \qquad \partial^\mu\partial_\mu \epsilon_\nu = 0. \tag{51}$$

Obviously, after the substitution $\tilde{h} = -2h_{44}$ contracted $\mu\nu$-equation (39) and the $\mu 4$-equation are satisfied identically in this gauge. Equation (35) for the $\mu\nu$-component reduces to an equation for a transverse-traceless tensor field $b_{\mu\nu}(x,y)$:

$$\frac{1}{2}(e^{2A(y)}\Box b_{\mu\nu} + \frac{\partial^2 b_{\mu\nu}}{\partial y^2}) - b_{\mu\nu}\left(2(A')^2 - A''\right) = 0. \tag{52}$$

This equation is absolutely analogous to equation (8) in the unstabilized RS1 model.

Scalar field equations follow from equations (37) and (38). It turns out that these equations are equivalent in the bulk and look much simpler, when written in terms of a new function g, which is related to h_{44} by $h_{44}(x,y) = e^{2A(y)}g(x,y)$:

$$e^{2A(y)}\Box g + 2A'g' - 2\frac{\phi''}{\phi'}g' - \frac{u^2\phi^2}{6M^3}g + g'' = 0. \tag{53}$$

This equation is similar to the one studied in [5]. The boundary conditions for this equation follow from (37) and (38) and look like

$$(\beta_1^2 + u)g' + \partial_\mu\partial^\mu g|_{y=+0} = 0, \tag{54}$$

$$(\beta_2^2 - u)g' - \partial_\mu\partial^\mu g|_{y=L-0} = 0. \tag{55}$$

Again it is easy to check that the massless mode of equation (52) is given by

$$b_{\mu\nu}(x,y) = e^{-2A(y)}\bar{h}_{\mu\nu}(x). \tag{56}$$

Residual gauge transformations (51) imply that the massless mode has only two physical degrees of freedom. Thus, the metric including only the massless tensor mode looks like

$$\begin{aligned}ds^2 = e^{-2A(y)}(\eta_{\mu\nu} + \bar{h}_{\mu\nu}(x))dx^\mu dx^\nu + dy^2 \equiv\\ \equiv e^{-2A(y)}\bar{g}_{\mu\nu}(x)dx^\mu dx^\nu + dy^2,\end{aligned} \tag{57}$$

$\bar{g}_{\mu\nu}(x)$ being given by (11).

Substituting this metric and the background field $\phi(y) = \phi_0 e^{-u|y|}$ into action (22), we arrive at the following expression:

$$S = 2M^3 \int_{-L}^{L} e^{-2A(y)} dy \int d^4x R_4(\bar{g}) \sqrt{-\bar{g}}, \tag{58}$$

the notations being the same as in (12). Integrating in the latter action over y, we get an effective four-dimensional action

$$S_{eff} = 2M_4^2 \int d^4x R_4(\bar{g}) \sqrt{-\bar{g}}. \tag{59}$$

The relation between the energy scales is now defined by the integral

$$M_4^2 = M^3 e^{-2c} \int_{-L}^{L} dy e^{-2k|y| - 2be^{-2u|y|}}, \tag{60}$$

which is explicitly given by

$$M_4^2 = \frac{M^3}{u} e^{-2c} (2b)^{-\frac{k}{u}} \left\{ \gamma\left(\frac{k}{u}, 2b\right) - \gamma\left(\frac{k}{u}, 2be^{-2uL}\right) \right\}. \tag{61}$$

Here b is defined in (30) and γ is the incomplete gamma function.

Again we got the standard gravitational action and therefore the field $\bar{h}_{\mu\nu}(x)$ is the graviton field. Its coupling to matter on the positive tension brane is defined by expression (61) with $c = -b$ and the coupling to matter on the negative tension brane is defined by the same expression with $c = -kL - be^{-2uL}$. Using the standard formulas for the incomplete gamma function, one can easily check that in the limit $b \to 0$ or $u \to \infty$ the former expression goes into (18), and the latter into (21) respectively.

One can analyse relation (61) in the case of physically motivated values of the parameters involved, but it turns out to be more convenient to use the integral representation (60) rather than the explicit formula in terms of incomplete gamma functions. Similar questions have been addressed in [5, 10]; moreover, in the latter paper the cosmological aspects of the stabilized warped brane models have been discussed.

1. The first obvious approximation is $b = \frac{\phi_0^2}{48M^3} \ll 1$, i.e. the backreaction of the scalar field on the RS metric is assumed to be small. This approximation was studied in [5]. Essentially it means that one can drop the term with b in (31) and retain it only in equation (53) for the scalar field. The boundary conditions for this field being rather complicated, in [5] it was suggested to use the limit $\beta_{1,2} \to \infty$. In this limit (54), (55) reduce to

$$g'|_{y=0} = g'|_{y=L} = 0. \tag{62}$$

The relations between the five- and four-dimensional energy scales are given by the same formulas (18) and (21) with the same values of the parameters M and k. The radion mass, i.e. the mass of the lowest scalar excitation, for an observer on the negative tension brane can be estimated as [5]

$$m_{rad}^2 = 32bu^2 e^{-2uL} \tag{63}$$

Recall that on the negative tension brane M, k and u can be of the order of 1 TeV. We also must have $kL \sim 35$ in order to get the correct value of Newton's constant. This means that to have the radion mass in the range of 100 GeV the ratio u/k has to be small. This observation suggests another approximation, namely $uL \ll 1$.

2. Assuming $uL \ll 1$, but keeping b arbitrary, we can expand the exponential in (31) to the first order in uL and get

$$A(y) \simeq (k - 2bu)|y| + c = \tilde{k}|y| + c, \quad \tilde{k} = (k - 2bu). \tag{64}$$

Here we restrict ourselves to considering only the negative tension brane, where our world is supposed to be. In this case the relation between the fundamental five-dimensional mass M and the Planck mass looks like

$$M_{Pl}^2 = M^3 \frac{e^{2\tilde{k}L} - 1}{\tilde{k}}. \tag{65}$$

To estimate the radion mass we have to solve equation (53) in this approximation. In so doing we also restrict ourselves to boundary conditions (62), which were used in [5].

Passing to a new variable

$$z = \frac{m}{\tilde{k}} e^{\tilde{k}y - \tilde{k}L} \tag{66}$$

brings equation (53) to the form of the Bessel equation, an approximate solution with boundary condition (62) at $y = 0$ being, up to normalization,

$$g(z) = z^{-(1+\frac{u}{\tilde{k}})} J_\alpha(z), \tag{67}$$

$$\alpha = \sqrt{(1 + \frac{u}{\tilde{k}})^2 + 16b\frac{u^2}{\tilde{k}^2}}. \tag{68}$$

The spectrum of scalar excitations is defined by boundary condition (62) at $y = L$, which gives for the lowest radion mass

$$m_{rad}^2 = \frac{4\tilde{k}^2(\alpha + 1)(\alpha - 1 - \frac{u}{\tilde{k}})}{(\alpha + 1 - \frac{u}{\tilde{k}})}. \tag{69}$$

This equation defines the radion mass for arbitrary values of b. If $b\frac{u^2}{\tilde{k}^2} \ll 1$, we can expand the square root in (68) to the leading order in this variable, which gives

$$m_{rad}^2 = 32bu^2. \tag{70}$$

This obviously coincides with (63) in the approximation under consideration.

3. The discussed scenarios are nevertheless very similar to the original RS1 model, where the four-dimensional worlds on different branes have hierarchically different energy scales. For certain values of parameters the stabilized model can lead to the same energy scale on both branes. This situation is described by equation

$$A(0) = A(L). \tag{71}$$

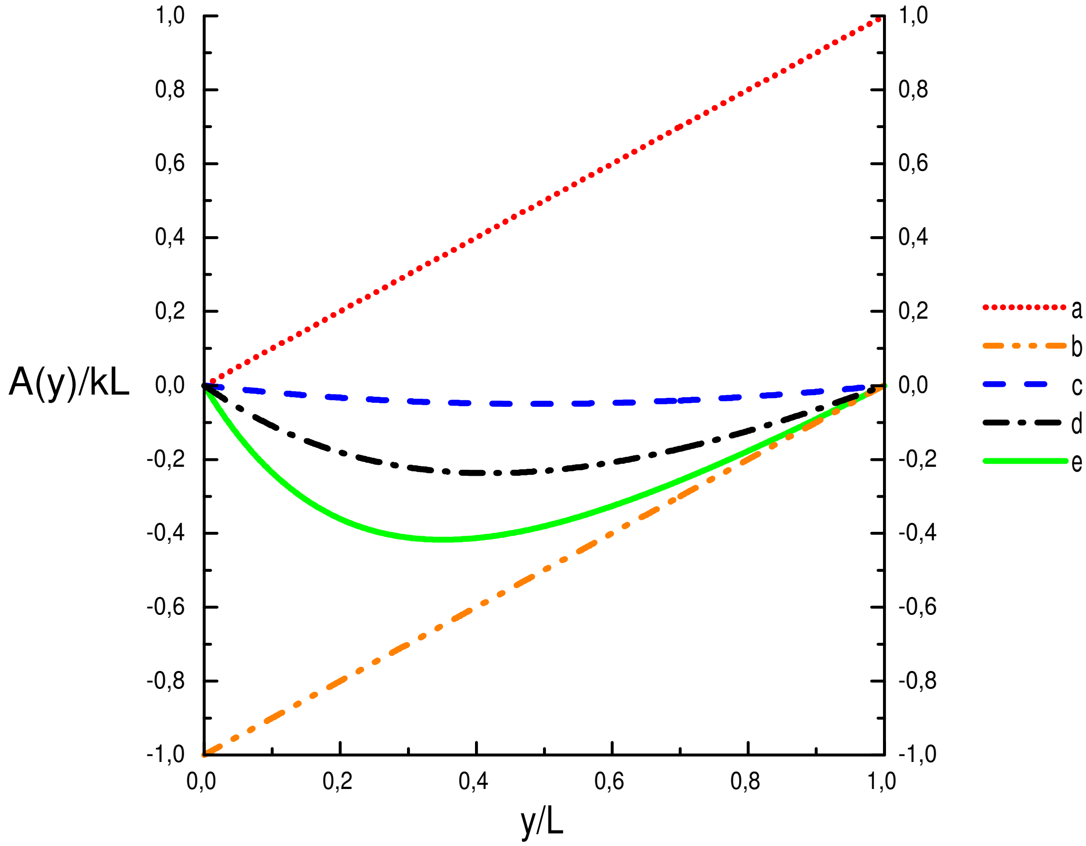

Figure 1: Plots of $\frac{A(y)}{kL}$ for different cases: a - RS1 model (observer on the positive tension brane); b - RS1 model (observer on the negative tension brane); c - "symmetric" case with $w = 0.2$; d - "symmetric" case with $w = 1$; e - "symmetric" case with $w = 2$.

From equation (71) one can express the b parameter in the following form

$$b = \frac{kL}{1 - e^{-2uL}}. \tag{72}$$

Now the function $A(y)$ (31) looks as follows

$$A(y) = kLf(\xi), \;\; f(\xi) = \left\{\xi - \frac{1 - e^{-2w\xi}}{1 - e^{-2w}}\right\}, \tag{73}$$

where $\xi = y/L$, $w = uL$. We have chosen the constant c in this equation so that $A(0) = A(L) = 0$, which leads to Galilean coordinates on both branes; one can easily see it from (73). From this point of view the case under consideration can be called "symmetric", although the function $A(y)$ (see Figure 1) is not symmetric with respect to $y/L = 0.5$.

The function $A(y)$ (31) is still rather complicated to solve the equation for the radion and gravity fields and find the masses of the lowest and excited states. So we will again use

the approximation $uL \ll 1$, which proved to be quite reasonable. In this case we expand the exponential in $A(y)$ to the second order in $2u|y| < 2uL \ll 1$ and get

$$A(y) = \tilde{k}|y| + 2b(uy)^2. \tag{74}$$

Equation (71) in this approximation gives a relation among the parameters of the model

$$uL = -\frac{\tilde{k}}{2bu}. \tag{75}$$

We also note that in this case the coordinates $\{x^\mu\}$ in (24) are Galilean on both branes.

Assuming b to be large enough, the integral relating the five- and four-dimensional energy scales can be estimated as

$$\int_{-L}^{L} e^{-2A(y)}dy \simeq \sqrt{\frac{\pi}{b}}\frac{e^{(uL)^2 b}}{u}. \tag{76}$$

The radion mass is again given by (70). It turns out that in this case it is possible to express the relation between the five- and four-dimensional energy scales in terms of the radion mass and the brane separation distance:

$$M_{Pl}^2 = 4\sqrt{2\pi}\frac{M^3}{m_{rad}}e^{\frac{m_{rad}^2 L^2}{32}}. \tag{77}$$

If one takes a rather small radion mass $m_{rad} \sim 100\,GeV$, so that the exponent in the (77) is of the order of unity, then one gets $M \sim 10^{12} - 10^{13} GeV$. This energy scale arises, for example, in the $SO(10)$ Grand Unification model with the $SO(10)$ broken first to $SU(4) \times SU(2) \times SU(2)$ as the intermediate scale, at which the quark-lepton $SU(4)$ symmetry is broken [11].

There is also a possibility in this scenario to have the fundamental five-dimensional energy scale of the order of $10TeV$, but in this case the radion must be very heavy, although the excitations of the tensor field remain in the TeV energy range.

Thus, we have shown that in the "bottom-up" approach, when the bulk in the stabilized brane models is considered from the viewpoint of an observer on "our" negative tension brane, the size of the bulk $L \sim TeV^{-1}$ appeared in a natural way. The lowest radion state in the most of the stabilized scenarios could be rather light in the range of $100GeV$, while its KK excitations, as well as KK excitations of the graviton are typically in the TeV range. In the most of the scenarios the fundamental gravitational multidimensional scale is in the TeV range. However, in the case of the "symmetric" warped scenario the scale could be significantly larger in the range of $10^{12} - 10^{13} GeV$.

The existence of the stabilized warped bulk with TeV^{-1} size opens up new possibilities for studying models with KK excitations of various bulk fields on "our" brane, which should naturally lie in the mass range of $L^{-1} \sim 1TeV$ leading to potentially interesting collider phenomenology.

We would like to make one more remark. For certain values of the parameters of the model the backreaction of the scalar field essentially amounts to a renormalization of the parameter k of the RS1 model, which is related to the bulk cosmological constant by $\Lambda = -24M^3k^2$ (6) and is responsible for the generation of the mass hierarchy. Explicitly, k is replaced by

$\tilde{k} = (k - 2bu)$ as in (64). An interesting fact is that the parameter k is also renormalized in the unstabilized RS1 model, when the Gauss-Bonnet term is added to the gravitational action [12, 13]. For small values of a dimensionless parameter α, which stands in front of the Gauss-Bonnet term together with the fundamental energy scale M, the renormalization of k looks like $k_{-} = k - \alpha k^3/M^2$. Depending on the sign of α, the variation of k can be either positive or negative, unlike only the negative variation in the case of the stabilized model. It would be interesting to find out, whether these two mechanisms could compensate each other and produce a stabilized model with the Gauss-Bonnet term and the Randall-Sundrum background metric along the lines of paper [14].

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